Entropy of Hidden Markov Models

A.C.C. van Wijk

supervisors

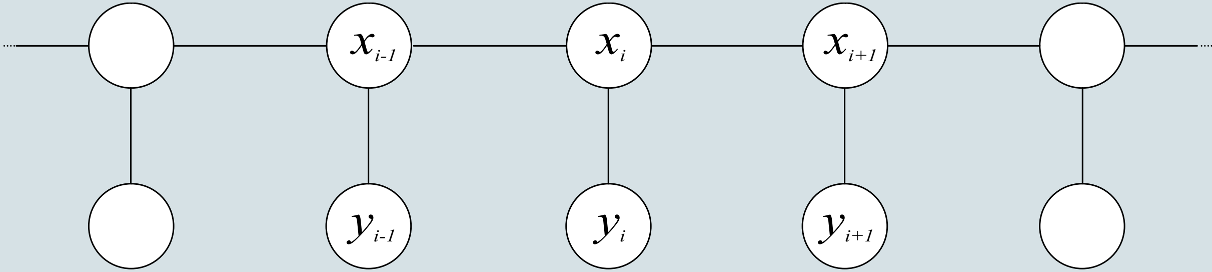
Philips Research: Dr. E.A. Verbitskiy, Dr. R. Rietman

TUE: Prof. dr. R.W. van der Hofstad, Prof. dr. ir. S.C. Borst

September 3, 2007

Hidden Markov Model

Stochastic process where transition probabilities depend on entire history of process.



Markov chain (X_n), hidden Markov model (Y_n),
Y_i only depends on X_i.

Binary Symmetric HMM

Markov chain (X_n) on $\{1,-1\}$, transition probabilities P,

Process (Y_n) on $\{1,-1\}$, Y_i only depends on X_i via Π.

(Y_n) is called *Binary Symmetric Hidden Markov Model*.

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}, \qquad \Pi = \begin{pmatrix} 1-\delta & \delta \\ \delta & 1-\delta \end{pmatrix}.$$

Example:

$$\begin{array}{lcccccccc} X_i: & \dots & -1 & 1 & 1 & -1 & 1 & -1 & \dots \\ Y_i: & \dots & -1 & 1 & -1 & -1 & -1 & -1 & \dots \end{array}$$

Entropy of random variable

Let U be a random variable. The entropy of U is given by

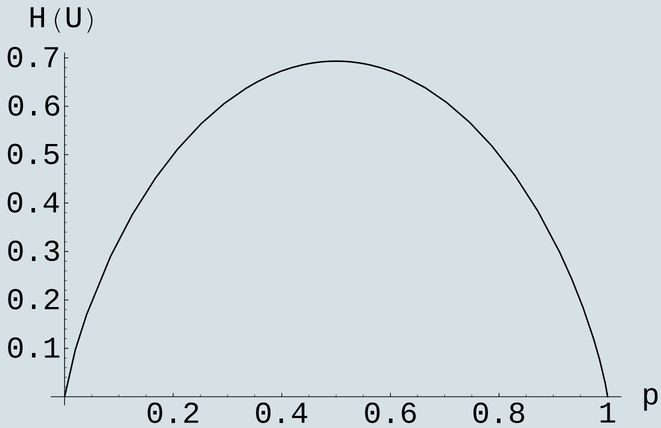
$$H(U) \;:=\; -\mathbb{E}[\,\log p(u)\,] \;=\; -\sum_{u \in \mathcal{U}} p(u) \log p(u).$$

Example:

$$U \quad = \quad \begin{cases} 1 & \text{with prob. } p, \\ -1 & \text{with prob. } 1-p, \end{cases}$$

then

$$H(U) = -p \log p \\ \qquad - (1-p)\log(1-p).$$



Entropy of stochastic process

Entropy of $(Y_0, \ldots, Y_n)$ for $n \to \infty$:

$$h_Y = \lim_{n\to\infty} \frac{1}{n+1} H(Y_0, \ldots, Y_n)$$

$$= \lim_{n\to\infty} H(Y_0 \mid Y_1, \ldots, Y_n),$$

where

$$H(Y_0 \mid Y_1, \ldots, Y_n) = -\mathbb{E}\big[\log \mathbb{P}[y_0 \mid y_1, \ldots, y_n]\big].$$

Entropy of hidden Markov models

- Series expansions for $H(Y_0 \mid Y_1, \ldots, Y_n)$ and $\mathbb{P}[Y_0 \mid Y_1, \ldots, Y_n]$
- Iterative calculation for $\mathbb{P}[Y_0 = 1 \mid Y_1, \ldots, Y_n]$
- Series expansion entropy h_Y in $\delta(1-\delta)$

Entropy of HMM

Power series expansion for entropy:

$$H(Y_0 \mid Y_1, \ldots, Y_n) = \sum_{k=0}^{\infty} f_k^{(n)}(p)\, \delta^k,$$

$$\lim_{n\to\infty} H(Y_0 \mid Y_1, \ldots, Y_n) = \sum_{k=0}^{\infty} f_k(p)\, \delta^k.$$

Stabilization of coefficients (Zuk et al., 2006):

$$f_k^{(n)} = f_k \quad \text{for} \quad n \geq \left\lceil \tfrac{k+1}{2} \right\rceil .$$

Conditional probability

Power series expansion for conditional probability:

$$\mathbb{P}[Y_0 \mid Y_1, \ldots, Y_n] = \sum_{k=0}^{\infty} F_k^{(n)}(p,y)\,\delta^k,$$

$$\lim_{n\to\infty} \mathbb{P}[Y_0 \mid Y_1, \ldots, Y_n] = \sum_{k=0}^{\infty} F_k(p,y)\,\delta^k.$$

Stabilization of coefficients:

$$F_k^{(n)} = F_k \quad \text{for} \quad n \geq k+1.$$

Conditional probability (2)

$$\mathbb{P}[Y_0 = 1 \mid Y_1 = 1, \ldots, Y_n = 1] = \sum_{k=0}^{\infty} F_k^{(n)}(p\,; 1, \ldots, 1)\, \delta^k,$$

n	$F_0^{(n)}$	$F_1^{(n)}$	$F_2^{(n)}$	$F_3^{(n)}$	$F_4^{(n)}$
0	$1/2$				
1	p	$2(1-2p)$	$-2(1-2p)$		
2	p	$\frac{1-2p}{p}$	$\frac{(1-2p)(3p-2)}{p^2}$	$\frac{-4(p-1)(1-2p)^2}{p^3}$	$\frac{-2(1-2p)^2(5p^2-10p+4)}{p^4}$
3	p	$\frac{1-2p}{p}$	$\frac{-(1-p)^2(1-2p)}{p^3}$	$\frac{-(1-2p)^2(2p^2-1)}{p^5}$	$\frac{-(1-2p)^2(5p^4-5p^2+1)}{p^7}$
4	p	$\frac{1-2p}{p}$	$\frac{-(1-p)^2(1-2p)}{p^3}$	$\frac{2(1-p)^2(1-2p)^2}{p^5}$	$\frac{-(1-2p)^2(p^4-4p^3+14p^2-14p+4)}{p^7}$
5	p	$\frac{1-2p}{p}$	$\frac{-(1-p)^2(1-2p)}{p^3}$	$\frac{2(1-p)^2(1-2p)^2}{p^5}$	$\frac{-(1-p)^2(1-2p)^2(p^2-10p+5)}{p^7}$
6	p	$\frac{1-2p}{p}$	$\frac{-(1-p)^2(1-2p)}{p^3}$	$\frac{2(1-p)^2(1-2p)^2}{p^5}$	$\frac{-(1-p)^2(1-2p)^2(p^2-10p+5)}{p^7}$

We have $F_k = F_k(p\,; y_0, \ldots, y_{k+1})$.

Unfortunately we can not find a general form for the F_k's.

Recurrence relations

Iterative calculation for

$$w_n(y_1, \ldots, y_n) := \mathbb{P}[Y_0 = 1 \mid Y_1 = y_1, \ldots, Y_n = y_n].$$

Define

$$W := \lim_{n\to\infty} w_n(y_1, \ldots, y_n).$$

Recurrence relations (2)

We have

$$\begin{aligned} w_n(1, y_2, \ldots, y_n) &= f_{+1}(w_{n-1}(y_2, \ldots, y_n)), \\ w_n(-1, y_2, \ldots, y_n) &= f_{-1}(w_{n-1}(y_2, \ldots, y_n)), \end{aligned}$$

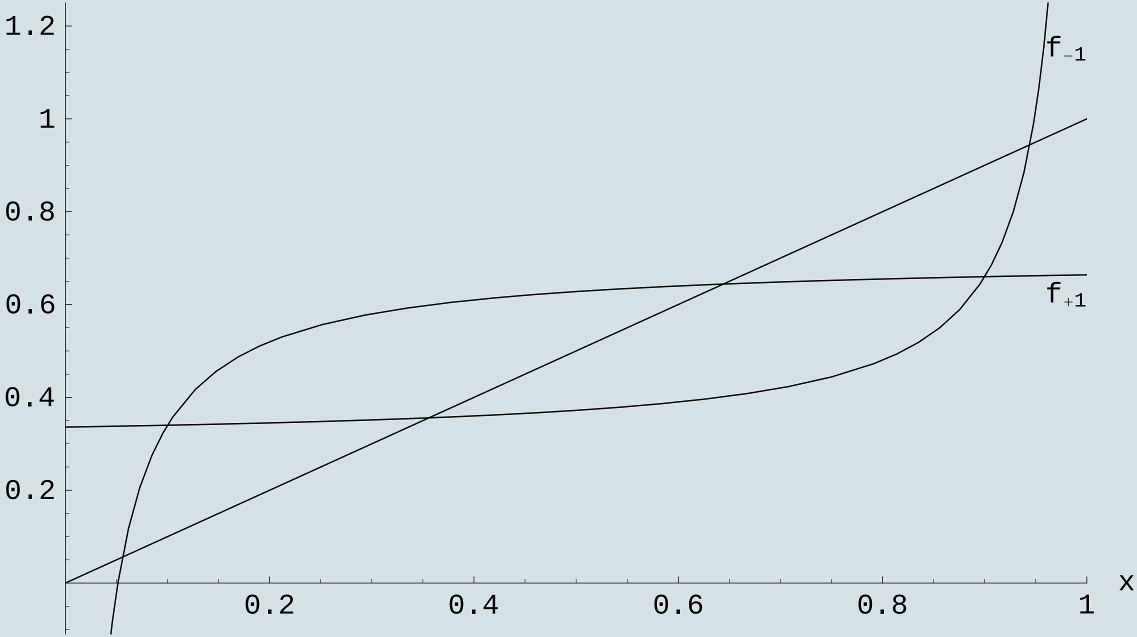
where

$$\begin{aligned} f_{+1}(x) &:= 1 - p - \frac{\delta(1-\delta)(1-2p)}{x}, \\ f_{-1}(x) &:= p + \frac{\delta(1-\delta)(1-2p)}{1-x}, \end{aligned}$$

so

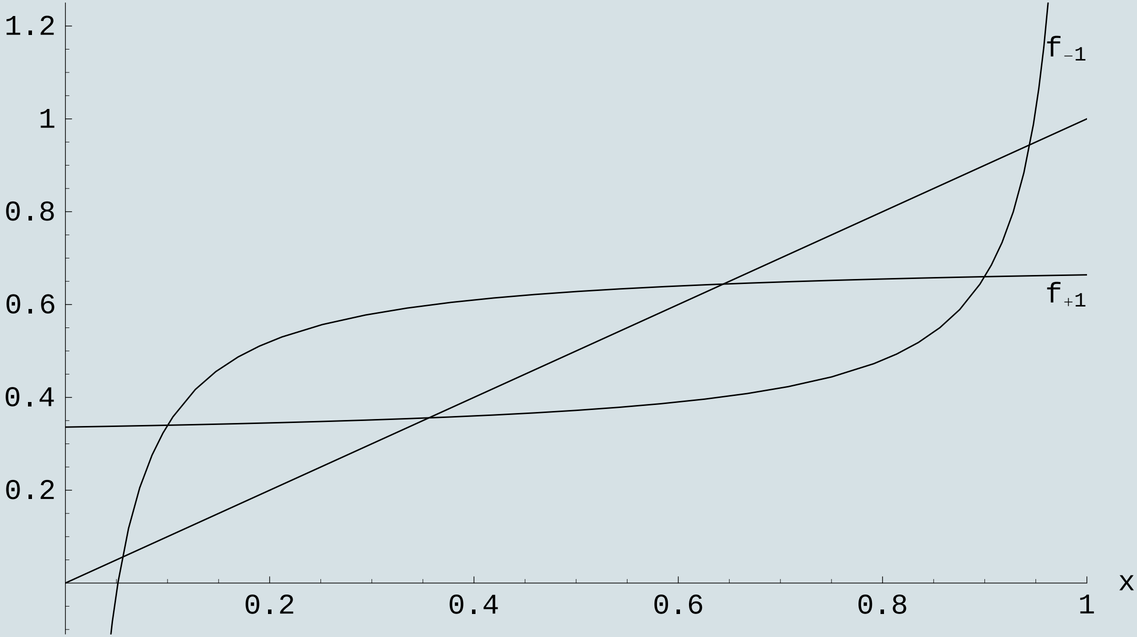
$$w_n(y_1, \ldots, y_n) = f_{y_1}\left(f_{y_2}\left(\ldots\left(f_{y_{n-1}}\left(f_{y_n}\left(\tfrac{1}{2}\right)\right)\right)\right)\right) .$$

Plot recurrence relations



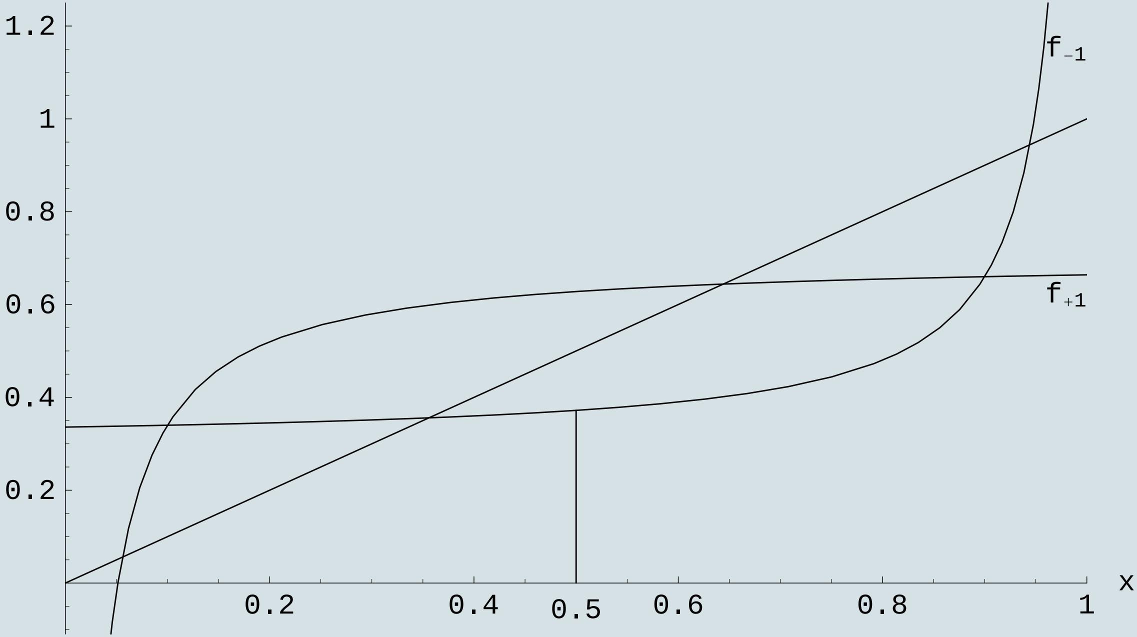
Iterating with f_{+1} and f_{-1}, for $p = 0.3$, $\delta = 0.1$.

Plot recurrence relations



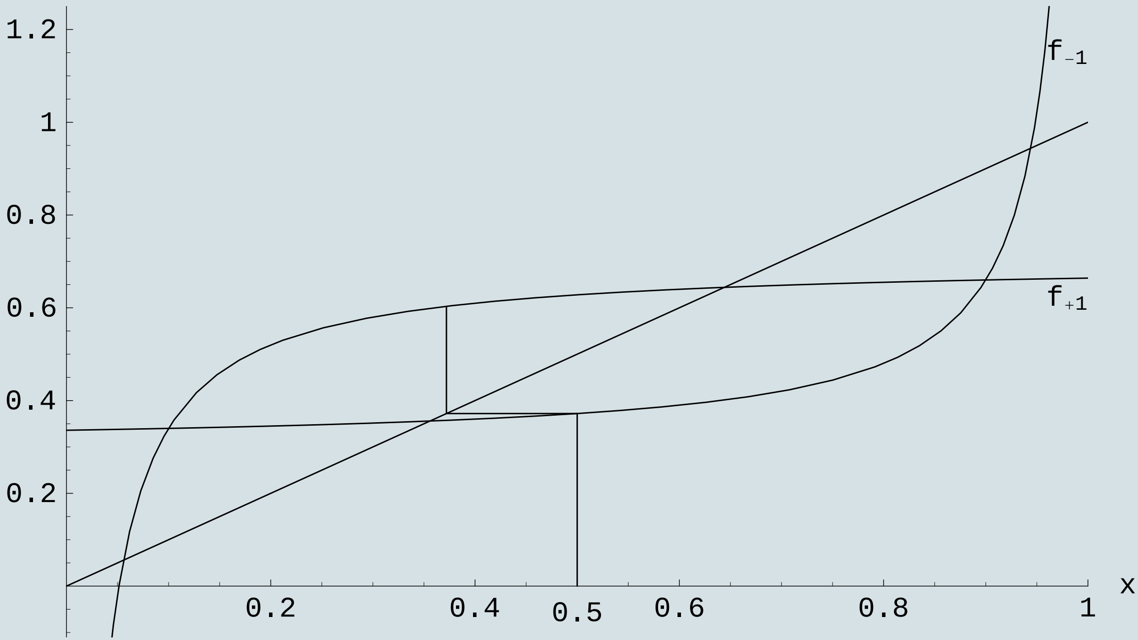
$$\mathbb{P}[Y_0 = 1 \mid Y_1 = 1, Y_2 = 1, Y_3 = -1] = w_n(+1, +1, -1)$$

Plot recurrence relations



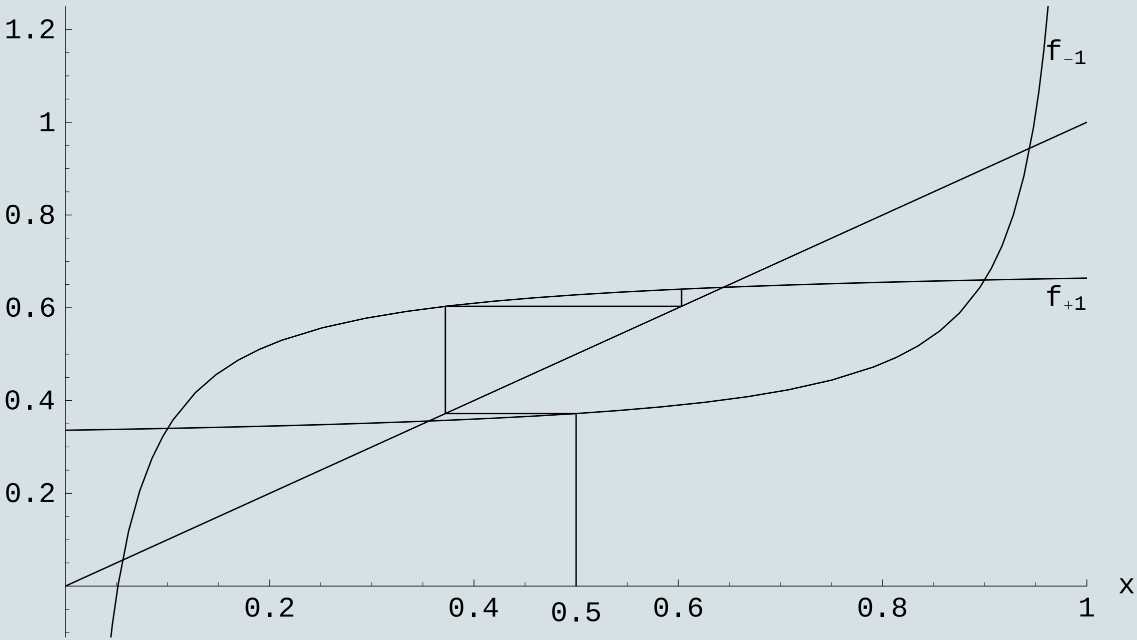
$$w_n(+1,+1,-1) = f_{+1}\left(f_{+1}\left(f_{-1}\left(1/2\right)\right)\right).$$

Plot recurrence relations



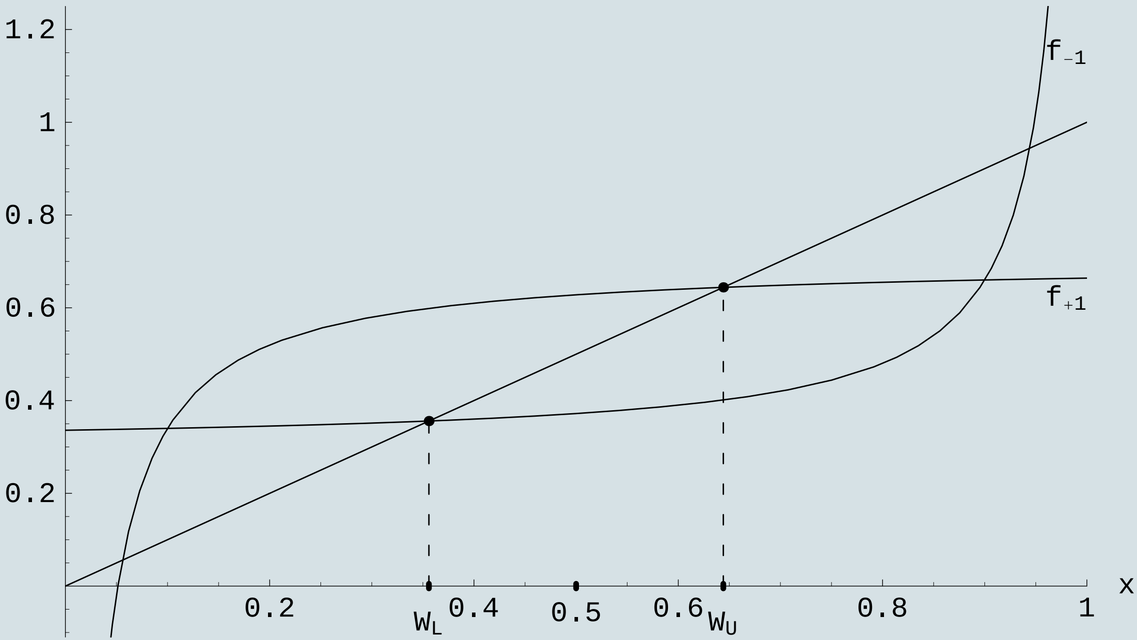
$$w_n(+1, +1, -1) = f_{+1}\left(f_{+1}\left(f_{-1}\left(1/2\right)\right)\right).$$

Plot recurrence relations



$$w_n(+1,+1,-1) = f_{+1}\left(f_{+1}\left(f_{-1}\left(1/2\right)\right)\right).$$

Plot recurrence relations



$$\text{Bounds}: \; W_L \leq w_n \leq W_U.$$

Proof recurrence relations

To be proven

$$
\begin{aligned}
w_n(1, y_2, \ldots, y_n) &= f_{+1}\left(w_{n-1}(y_2, \ldots, y_n)\right) \\
&= 1 - p - \frac{\delta(1-\delta)(1-2p)}{w_{n-1}(y_2, \ldots, y_n)}.
\end{aligned}
$$

Proof recurrence relations

To be proven

$$\begin{aligned} w_n(1, y_2, \ldots, y_n) &= f_{+1}\left(w_{n-1}(y_2, \ldots, y_n)\right) \\ &= 1 - p - \frac{\delta(1-\delta)(1-2p)}{w_{n-1}(y_2, \ldots, y_n)}. \end{aligned}$$

Proof

$$\begin{aligned} w_n(1, y_2, \ldots, y_n) &= \mathbb{P}[Y_0 = 1 \mid Y_1 = 1, Y_2, \ldots, Y_n] \\ &= \frac{\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid Y_2, \ldots, Y_n]}{\mathbb{P}[Y_1 = 1 \mid Y_2, \ldots, Y_n]} \end{aligned}$$

Proof recurrence relations (2)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid Y_2, \ldots, Y_n]}{\mathbb{P}[Y_1 = 1 \mid Y_2, \ldots, Y_n]}$$

Proof recurrence relations (2)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid Y_2, \ldots, Y_n]}{\mathbb{P}[Y_1 = 1 \mid Y_2, \ldots, Y_n]}$$

$$= \Big(\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid X_1 = 1]\, \mathbb{P}[X_1 = 1 \mid Y_2, \ldots, Y_n]$$

$$+ \mathbb{P}[Y_0 = 1, Y_1 = 1 \mid X_1 = -1]\, \mathbb{P}[X_1 = -1 \mid Y_2, \ldots, Y_n]\Big) /$$

$$(\ldots)$$

Proof recurrence relations (2)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid Y_2, \ldots, Y_n]}{\mathbb{P}[Y_1 = 1 \mid Y_2, \ldots, Y_n]}$$

$$= \Big(\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid X_1 = 1]\, \mathbb{P}[X_1 = 1 \mid Y_2, \ldots, Y_n] + \mathbb{P}[Y_0 = 1, Y_1 = 1 \mid X_1 = -1]\, \mathbb{P}[X_1 = -1 \mid Y_2, \ldots, Y_n]\Big) / \Big(\mathbb{P}[Y_1 = 1 \mid X_1 = 1]\, \mathbb{P}[X_1 = 1 \mid Y_2, \ldots, Y_n] + \mathbb{P}[Y_1 = 1 \mid X_1 = -1]\, \mathbb{P}[X_1 = -1 \mid Y_2, \ldots, Y_n]\Big)$$

Proof recurrence relations (2)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid Y_2, \ldots, Y_n]}{\mathbb{P}[Y_1 = 1 \mid Y_2, \ldots, Y_n]}$$

$$= \Big(\mathbb{P}[Y_0 = 1, Y_1 = 1 \mid X_1 = 1]\,\mathbb{P}[X_1 = 1 \mid Y_2, \ldots, Y_n] + \mathbb{P}[Y_0 = 1, Y_1 = 1 \mid X_1 = -1]\,\mathbb{P}[X_1 = -1 \mid Y_2, \ldots, Y_n]\Big) / \Big(\mathbb{P}[Y_1 = 1 \mid X_1 = 1]\,\mathbb{P}[X_1 = 1 \mid Y_2, \ldots, Y_n] + \mathbb{P}[Y_1 = 1 \mid X_1 = -1]\,\mathbb{P}[X_1 = -1 \mid Y_2, \ldots, Y_n]\Big)$$

Let $q := \mathbb{P}[X_1 = 1 \mid Y_2, \ldots, Y_n]$.

Proof recurrence relations (3)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{q(1-\delta)\left((1-p)(1-\delta)+p\delta\right)+(1-q)\delta\left(p(1-\delta)+(1-p)\delta\right)}{q(1-\delta)+(1-q)\delta}$$

Proof recurrence relations (3)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{q(1-\delta)\left((1-p)(1-\delta)+p\delta\right)+(1-q)\delta\left(p(1-\delta)+(1-p)\delta\right)}{q(1-\delta)+(1-q)\delta}$$

$$= \frac{(1-p)\left(q(1-\delta)+(1-q)\delta\right)+\delta(1-\delta)(1-2p)}{q(1-\delta)+(1-q)\delta}$$

Proof recurrence relations (3)

$$w_n(1, y_2, \ldots, y_n)$$

$$= \frac{q(1-\delta)\left((1-p)(1-\delta)+p\delta\right)+(1-q)\delta\left(p(1-\delta)+(1-p)\delta\right)}{q(1-\delta)+(1-q)\delta}$$

$$= \frac{(1-p)\left(q(1-\delta)+(1-q)\delta\right)+\delta(1-\delta)(1-2p)}{q(1-\delta)+(1-q)\delta}$$

$$= 1-p-\frac{\delta(1-\delta)(1-2p)}{\mathbb{P}[Y_1=1 \mid Y_2, \ldots, Y_n]},$$

so

$$w_n(1, y_2, \ldots, y_n) = f_{+1}\left(w_{n-1}(y_2, \ldots, y_n)\right). \quad \square$$

Simulation

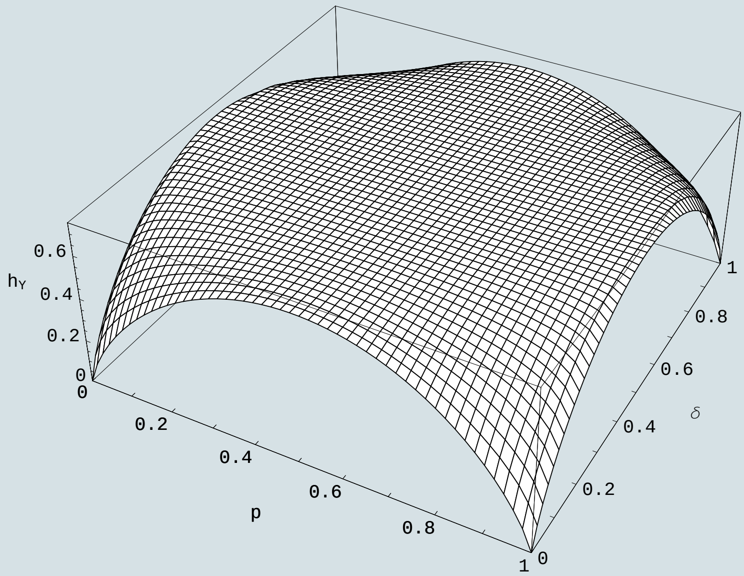
```
{n = 100, x = 0.5, sum = 0.}

For[i = 0, i <= n, i++,

  sum = sum +
    (-x Log[x] - (1-x) Log[1-x]);

  If[Random[] < x,
     x = f+[x],
     x = f-[x]
  ];

];
sum/(n+1)
```



Series expansion entropy

We have

$$w_n = \begin{cases} f_{+1}(w_{n-1}) & \text{w.p. } w_{n-1}, \\ f_{-1}(w_{n-1}) & \text{w.p. } 1 - w_{n-1}. \end{cases}$$

Let

$$h(p) = -p \log p - (1-p)\log(1-p),$$

then

$$\begin{aligned} h_Y &= \lim_{n\to\infty} \mathbb{E}[h(w_n)] \\ &= \lim_{n\to\infty} \mathbb{E}[\, w_{n-1}\, h(f_{+1}(w_{n-1})) + (1 - w_{n-1})\, h(f_{-1}(w_{n-1}))\,] \\ &= \mathbb{E}[\, W\, h(f_{+1}(W)) + (1-W)\, h(f_{-1}(W))\,]. \end{aligned}$$

Series expansion entropy (2)

Let $\zeta := \delta(1-\delta)$, then

$$\begin{aligned} h_Y &= \mathbb{E}[W\, h(f_+(W)) + (1-W)\, h(f_-(W))] \\ &= \mathbb{E}\Big[W\, h\left(1 - p - \frac{\zeta(1-2p)}{W}\right) + (1-W)\, h\left(p + \frac{\zeta(1-2p)}{1-W}\right)\Big] \\ &= \mathbb{E}\Big[W\, h\left(p + \frac{\zeta(1-2p)}{W}\right) + (1-W)\, h\left(p + \frac{\zeta(1-2p)}{1-W}\right)\Big], \end{aligned}$$

as $h(x) = h(1-x)$.

Series expansion entropy (3)

Expand h in terms of ζ around $\zeta = 0$:

$$h_Y \;=\; \mathbb{E}\Big[W\,h\left(p + \frac{\zeta(1-2p)}{W}\right) + (1-W)\,h\left(p + \frac{\zeta(1-2p)}{1-W}\right)\Big]$$

Series expansion entropy (3)

Expand h in terms of ζ around $\zeta = 0$:

$$
\begin{aligned}
h_Y &= \mathbb{E}\Big[W\,h\left(p + \frac{\zeta(1-2p)}{W}\right) + (1-W)\,h\left(p + \frac{\zeta(1-2p)}{1-W}\right)\Big] \\
&= \mathbb{E}\Big[W\,h(p) + \ldots \\
&\qquad + \zeta\,(1-2p)\,h'(p) + \ldots \\
&\qquad + \zeta^2\,\frac{(1-2p)^2}{2W}\,h''(p) + \ldots\Big].
\end{aligned}
$$

Series expansion entropy (3)

Expand h in terms of ζ around $\zeta = 0$:

$$
\begin{aligned}
h_Y &= \mathbb{E}\Big[W\, h\left(p + \frac{\zeta(1-2p)}{W}\right) + (1-W)\, h\left(p + \frac{\zeta(1-2p)}{1-W}\right)\Big] \\
&= \mathbb{E}\Big[W\, h(p) + (1-W)\, h(p) \\
&\qquad + \zeta\,(1-2p)\, h'(p) + \zeta\,(1-2p)\, h'(p) \\
&\qquad + \zeta^2\, \frac{(1-2p)^2}{2W}\, h''(p) + \zeta^2\, \frac{(1-2p)^2}{2(1-W)}\, h''(p) + \ldots\Big].
\end{aligned}
$$

Series expansion entropy (4)

This gives

$$h_Y = h(p) + 2\,h'(p)\,(1-2p)\,\zeta + \sum_{k=2}^{\infty} \frac{h^{(k)}(p)}{k!}\,(1-2p)^k\,\zeta^k\,\mathbb{E}\left[\frac{1}{W^{k-1}} + \frac{1}{(1-W)^{k-1}}\right].$$

We also have

$$\mathbb{E}\left[\frac{1}{W^k} + \frac{1}{(1-W)^k}\right] = \sum_{n=0}^{\infty} R_{k,n}(p)\,\zeta^n,$$

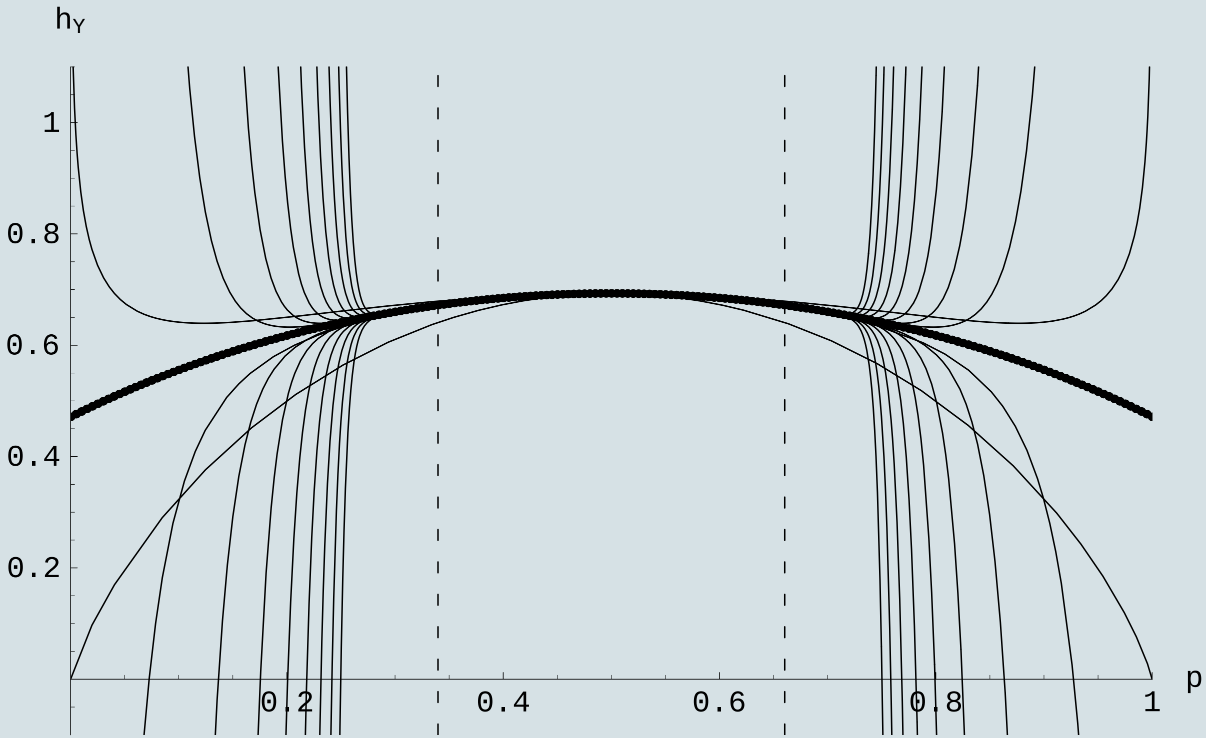
where the $R_{k,n}(p)$ can be iteratively calculated.

Series expansion entropy (5)

This gives a power series expansion for h_Y expanded in ζ around $\zeta = 0$:

$$h_Y = h(p) + 2\,h'(p)\,(1-2p)\,\zeta + \sum_{n=2}^{\infty}\left[\sum_{k=2}^{\infty} \frac{h^{(k)}(p)}{k!}\,(1-2p)^k\,R_{k-1,n-k}(p)\right]\zeta^n.$$

Plot series expansion entropy



$$\delta = 0.1$$

Summary

Main results

- Stabilization of coefficients $\mathbb{P}[Y_0 \mid Y_1, \ldots, Y_n]$
- Power series expansion for h_Y