Head-of-line processor sharing: optimal control

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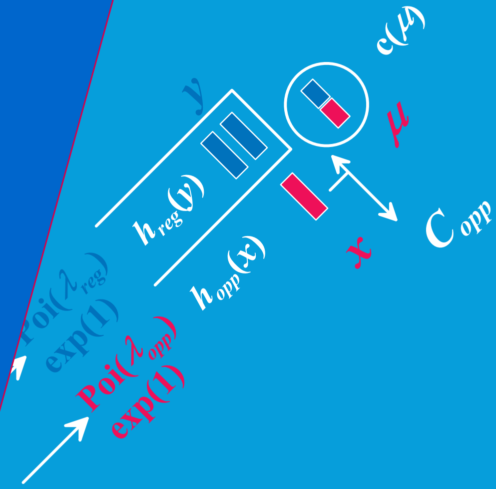

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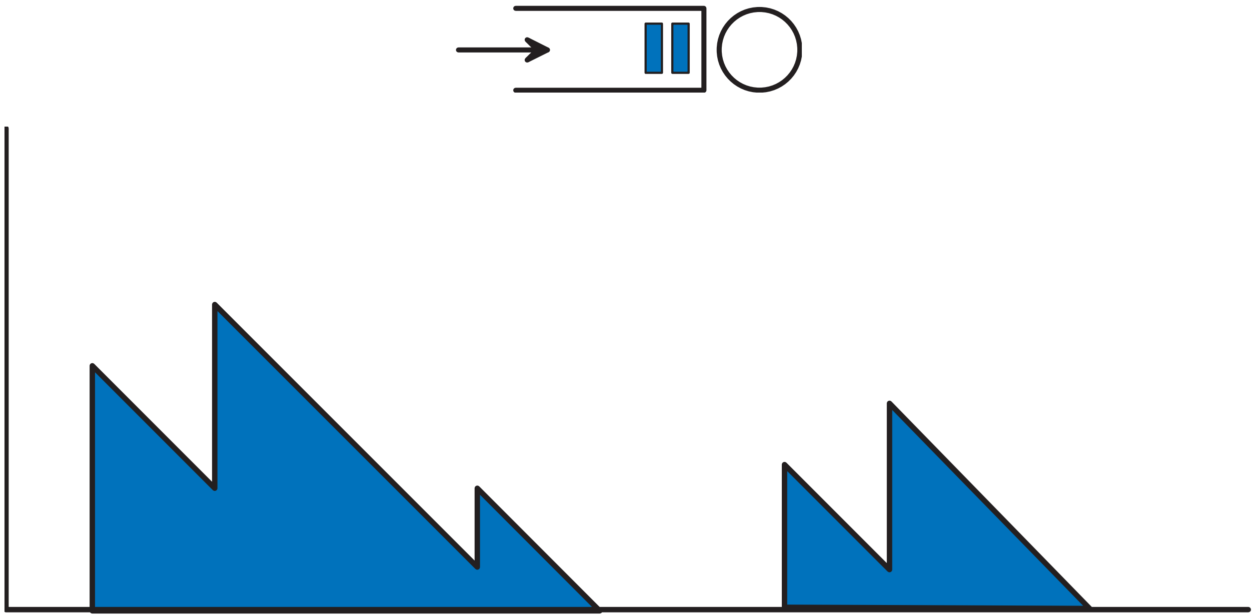
Introduction

Workload in $M/M/1$ queue



Application: e.g. production system

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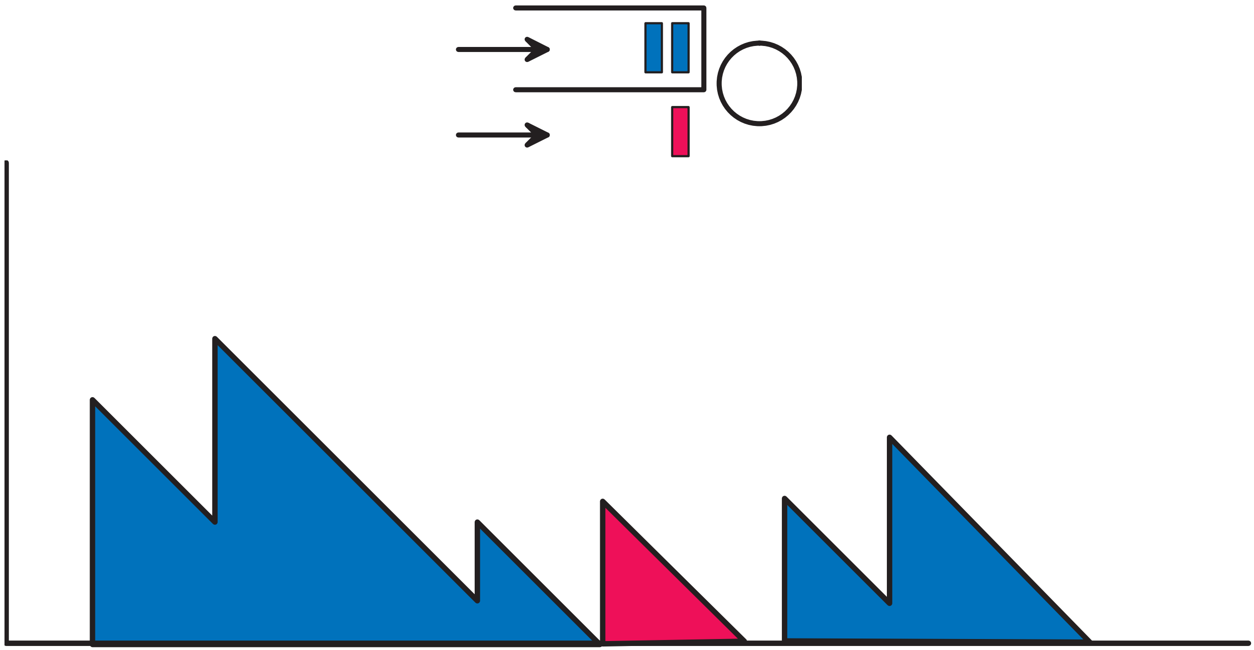
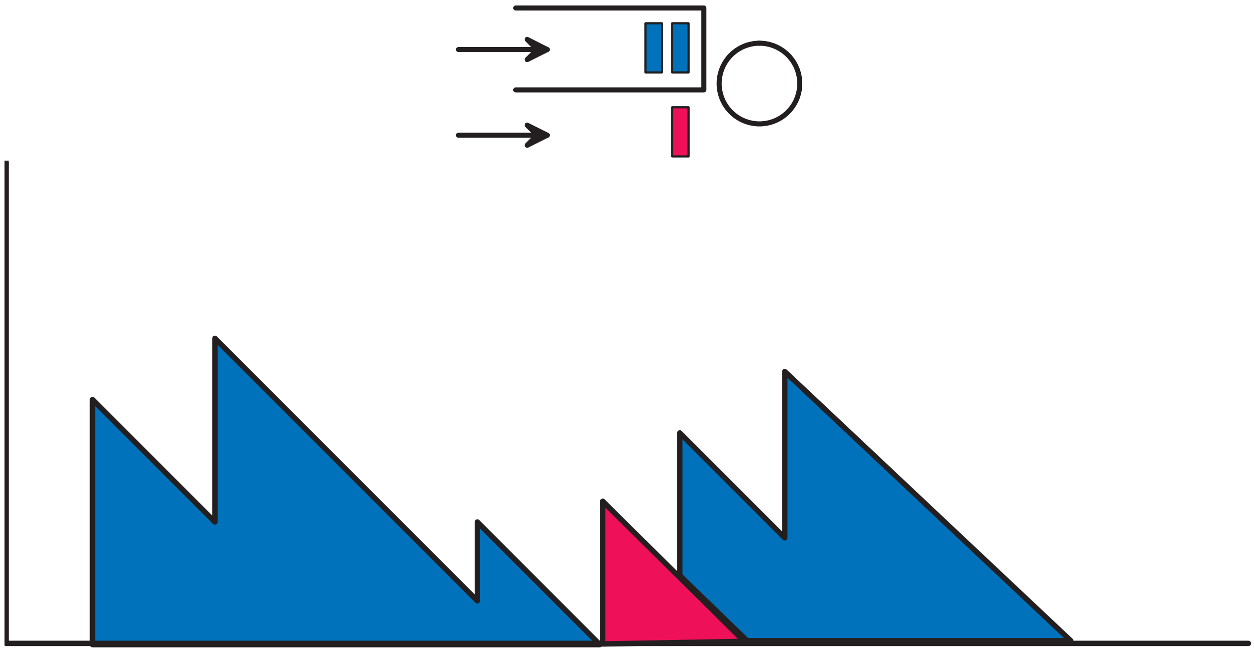
Workload in $M/M/1$ queue



Extra revenue for serving opportunity customers

Workload in $M/M/1$ queue

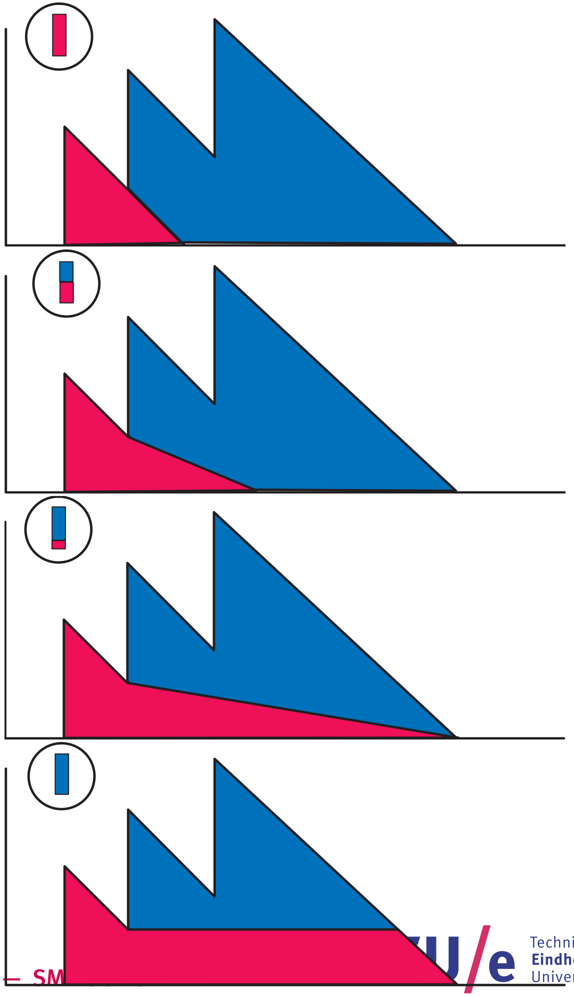


However, waiting times (and hence costs) of regular customers increase

Introduction

Workload in $M/M/1$ queue

- Regular and opportunity customers
- Head-of-line processor sharing



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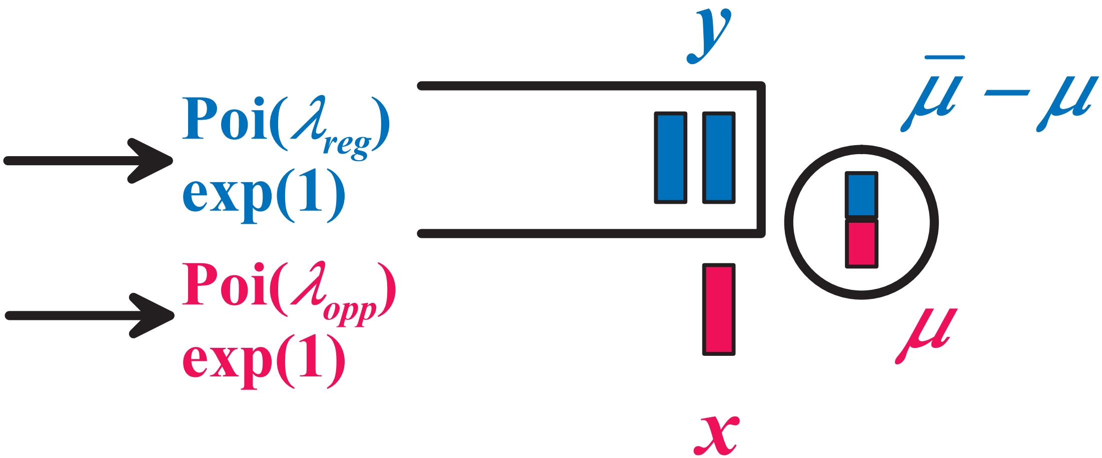
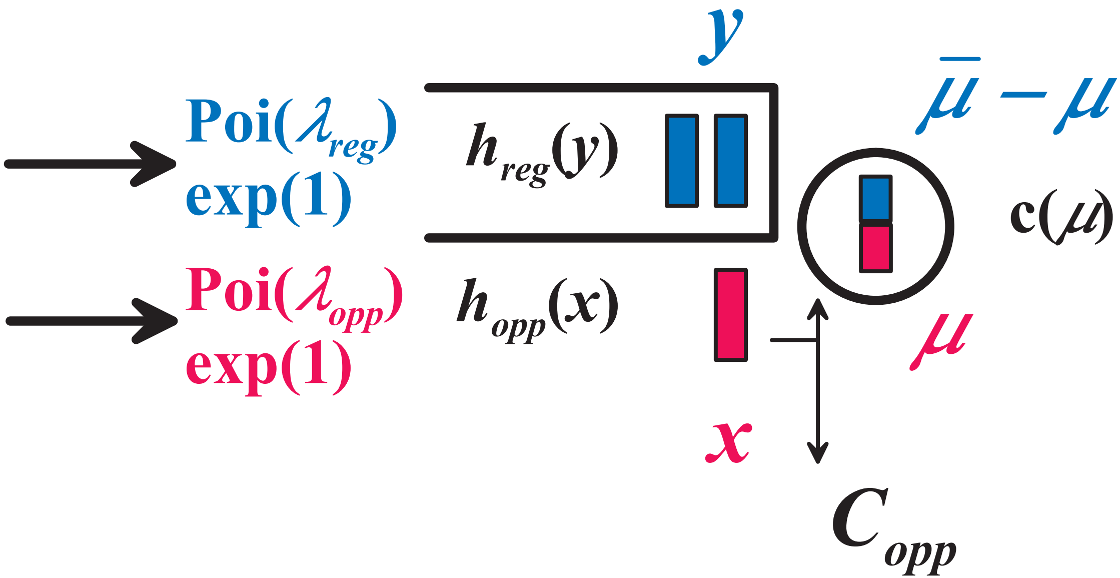
Model



Research Question: What is optimal control policy?

Model



Research Question: What is optimal control policy?

Markov Decision Problem

Value function:

$$
\begin{aligned}
&V_{n+1}(x,y)\\
&= h_{opp}(x) + h_{reg}(y) + \frac{1}{\lambda_{opp} + \lambda_{reg} + \bar{\mu} + \alpha}\\
&\quad \Big(\lambda_{reg} V_n(x, y+1)\\
&\quad + \lambda_{opp} \left\{ \begin{array}{ll} \min\{V_n(x+1,y), V_n(x,y) + C_{opp}\} & \text{if } x = 0 \\ V_n(x,y) + C_{opp} & \text{if } x = 1 \end{array} \right\}\\
&\quad + \min_{\mu \in [0,\bar{\mu}]} \left\{ c'(\mu) + \mu V_n((x-1)^+, y) + (\bar{\mu} - \mu) V_n(x, (y-1)^+) \right\} \Big),\\
&V_0 \equiv 0.
\end{aligned}
$$

Structural properties of value function imply optimal control policy structure.

Value function

Structural property: Multimodularity (MM, cf. Hajek, 1985)

Multimodularity (for 2 dimensions):

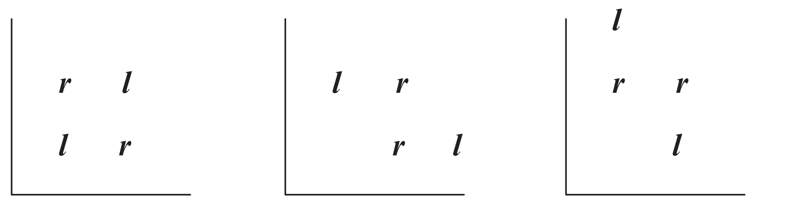
Supermodularity: $f(x, y) + f(x+1, y+1) \geq f(x+1, y) + f(x, y+1)$,
Superconvexity(1,2): $f(x+2, y) + f(x, y+1) \geq f(x+1, y) + f(x+1, y+1)$,
Superconvexity(2,1): $f(x, y+2) + f(x+1, y) \geq f(x, y+1) + f(x+1, y+1)$,

which implies:
Convexity(1): $f(x, y) + f(x+2, y) \geq 2f(x+1, y)$,
Convexity(2): $f(x, y) + f(x, y+2) \geq 2f(x, y+1)$.

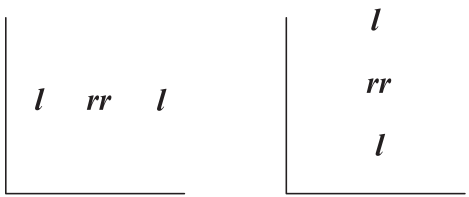
Value function

Structural property: Multimodularity (MM, cf. Hajek, 1985)

Multimodularity (for 2 dimensions): $l \geqslant r$



Supermodularity Superconvexity(1,2) Superconvexity(2,1)



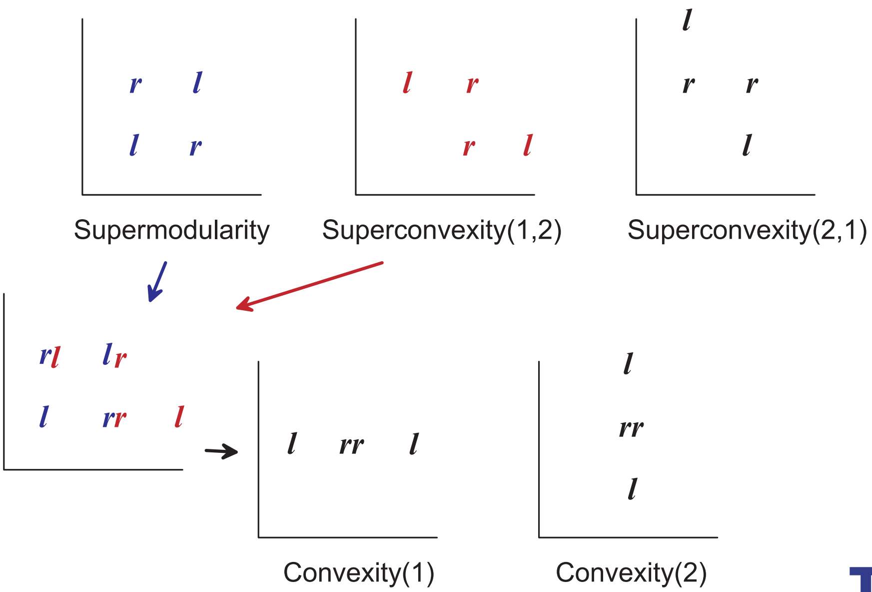
Convexity(1) Convexity(2)

Value function

Structural property: Multimodularity (MM, cf. Hajek, 1985)

Multimodularity (for 2 dimensions):

$l \geq r$



Value function

Prove that V_n is MM by induction on n:

- $V_0 \equiv 0$ is MM,
- Assume V_n is MM, show that V_{n+1} is MM.

Recall:

$$
\begin{aligned}
& V_{n+1}(x, y) \\
& = h_{opp}(x) + h_{reg}(y) + \frac{1}{\lambda_{opp} + \lambda_{reg} + \bar{\mu} + \alpha} \\
& \quad \Big(\lambda_{reg} V_n(x, y+1) \\
& \quad + \lambda_{opp} \begin{Bmatrix} \min\{V_n(x+1, y), V_n(x, y) + C_{opp}\} & \text{if } x = 0 \\ V_n(x, y) + C_{opp} & \text{if } x = 1 \end{Bmatrix} \\
& \quad + \min_{\mu \in [0, \bar{\mu}]} \Big\{ c'(\mu) + \mu V_n((x-1)^+, y) + (\bar{\mu} - \mu) V_n(x, (y-1)^+) \Big\} \Big),
\end{aligned}
$$

Value function

Rewrite value function using event operators (cf. Koole, 2006)

$$V_{n+1}(x, y) = T_{\text{costs}}\Big(T_{\text{unif}}\big(T_{CA(1)}V_n(x, y),\; T_{A(2)}V_n(x, y),\; \tilde{T}_{CTD(1)}V_n(x, y)\big)\Big)$$

Event operators:

- Arrivals of opportunity customers (decision!)

$$T_{CA(1)}f(x, y) = \min\{V_n(x+1, y), V_n(x, y) + C_{opp}\}$$

- Arrivals of regular customers

$$T_{A(2)}f(x, y) = V_n(x, y+1)$$

Value function

Rewrite value function using event operators (cf. Koole, 2006)

$$V_{n+1}(x, y)$$
$$= T_{\text{costs}}\Big(T_{\text{unif}}\big(T_{CA(1)}V_n(x, y),\ T_{A(2)}V_n(x, y),\ \tilde{T}_{CTD(1)}V_n(x, y)\big)\Big)$$

Event operators (ctd.):

- Service completions (decision!)

$$\tilde{T}_{CTD(1)} f(x, y) = \min_{\mu \in [0,1]} \Big\{ c(\mu) + \mu V_n((x-1)^+, y) + (1-\mu) V_n(x, (y-1)^+) \Big\}$$

Value function

Rewrite value function using event operators (cf. Koole, 2006)

$$V_{n+1}(x,y) = T_{\text{costs}}\Big(T_{\text{unif}}\big(T_{CA(1)}V_n(x,y),\ T_{A(2)}V_n(x,y),\ \tilde{T}_{CTD(1)}V_n(x,y)\big)\Big)$$

Event operators (ctd.):

- Costs

$$T_{\text{costs}} f(x,y) = h_{opp}(x) + h_{reg}(y) + f(x,y)$$

- Uniformization

$$T_{\text{unif}}(f_1, f_2, f_3)(x,y) = \frac{\lambda_{opp} f_1(x,y) + \lambda_{reg} f_2(x,y) + \bar{\mu} f_3(x,y)}{\lambda_{opp} + \lambda_{reg} + \bar{\mu} + \alpha}$$

Analysis

Value function

Use known results for operators (Koole, 2006):

- V_n is MM $\Rightarrow$ $T_{CA(1)}V_n$ is MM
- V_n is MM $\Rightarrow$ $T_{A(2)}V_n$ is MM
- V_n is MM $\Rightarrow$ $T_{\text{unif}}V_n$ is MM
- V_n is MM $\Rightarrow$ $T_{\text{costs}}V_n$ is MM

And $\tilde{T}_{CTD(1)}$?

Value function

Compare $\tilde{T}_{CTD(1)}$ to departure operator in tandem queue:

$$\tilde{T}_{CTD(1)} f(x,y) = \min_{\mu \in [0,1]} \left\{ c(\mu) + \mu V_n((x-1)^+, y) + (1-\mu) V_n(x, (y-1)^+) \right\}$$

$$T_{CTD(1)} f(x,y) = \min_{\mu \in [0,1]} \left\{ c(\mu) + \mu V_n((x-1)^+, y+1) + (1-\mu) V_n(x, y) \right\}$$

Known result: V_n is MM $\Rightarrow$ $T_{CTD(1)} V_n$ is MM

- Transformation: $y \to y - 1$
- check $x = 0,\ y = 0$ and $x > 0,\ y = 0$

Then:

- V_n is MM $\Rightarrow$ $\tilde{T}_{CTD(1)} V_n$ is MM

Main result

Value function is MM. Implies optimal policy structure.

Optimal policy structure for a head-of-line processor sharing model, with adjustable weights and two types of customers:

- Threshold T for admitting opportunity customer: accept if $y \leq T$, reject it otherwise.
- The optimal server speed dedicated to the opportunity customer is a monotone deceasing function in x.

Example 1: $c(\mu) \equiv 0$

$\lambda_{reg} = 3$, $\lambda_{opp} = 1$, $C_{opp} = 8$, $\bar{\mu} = 10$, $h_{opp}(x) = x$ and $h_{reg}(y) = 0.05\, y^2$ if $y < 20$; $h_{reg}(y) = 100\, y$ otherwise.

Optimal policy accept/reject opportunity customer:

$x \backslash y$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Threshold policy with $T = 6$.

The optimal fraction $\mu \in [0, 1]$ of server speed for opp. customer:

$x \backslash y$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0

So, opportunity customers either gets full attention of the server, or no attention at all, with threshold $y = 10$.

Example 1: $c(\mu) \equiv 0$

- If $c(\mu) \equiv 0$ (or: constant) also threshold policy for optimal server speed for opp. customer.
- Because $\tilde{T}_{CTD(1)}$ minimizes a linear function:

$$\begin{aligned}
&\tilde{T}_{CTD(1)} f(x, y) \\
&= \min_{\mu \in [0,1]} \Big\{ c(\mu) + \mu V_n((x-1)^+, y) \\
&\qquad\qquad + (1-\mu) V_n(x, (y-1)^+) \Big\}
\end{aligned}$$

Example 2: $c(\mu) \neq 0$

Same parameters, now:

$$c(\mu) = \begin{cases} 0 & \text{if } 0 \leq \mu < 0.25; \\ 0.5 & \text{if } 0.25 \leq \mu < 0.50; \\ 1 & \text{if } 0.50 \leq \mu < 0.75; \\ 1.5 & \text{if } 0.75 \leq \mu \leq 1. \end{cases}$$

Threshold $T = 5$ for accepting opportunity customers.

The optimal fraction $\mu \in [0, 1]$ of the service speed for the opp. customer:

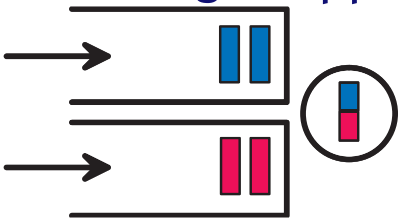
$x \backslash y$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0.75	0.25	0.25	0.25	0.25	0	0	0

Indeed monotone decreasing.

Extensions

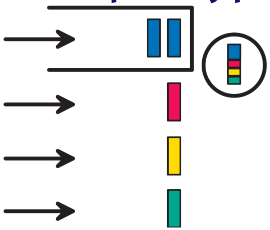
Model extensions

- Accept or reject regular customers
- Queueing of opportunity customers



Further research

- Steady state probability distribution
- Total service rate increases or decreases when the server divides its attention to two customers
- Multiple types of opportunity customers



Conclusion

Optimal policy structure for a head-of-line processor sharing model, with adjustable weights and two types of customers.

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