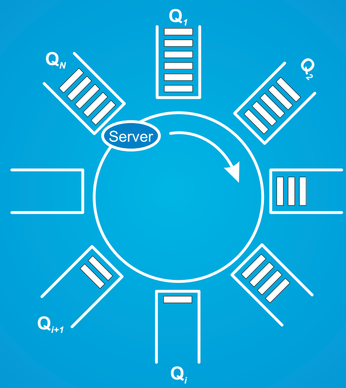
EURANDOM

A Polling Model with Smart Customers

Sandra van Wijk

A.C.C.v.Wijk@tue.nl

Joint work with Marko Boon, Ivo Adan, and Onno Boxma



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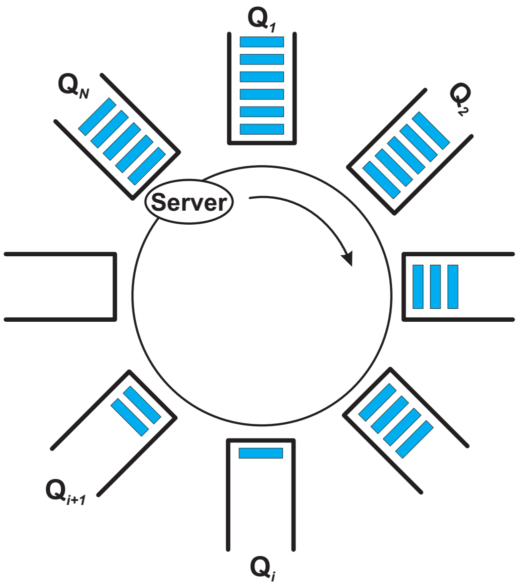
Introduction

Classical Polling System

- N queues,
- Arrivals at Q_i: Poisson(λ_i),
- Service time at Q_i: B_i,
- Load at Q_i: $\rho_i = \lambda_i \mathsf{E}(B_i)$,
- Visit time at Q_i: V_i,
- Switch-over time from Q_i to Q_{i+1}: S_i,
- Waiting time customer at Q_i: W_i.



Cycle:

$$V_1 - S_1 - V_2 - S_2 - \ldots - V_N - S_N.$$

Applications: telecommunication, repairman, production, etc.

Smart Customers

cf. Boxma (1994)

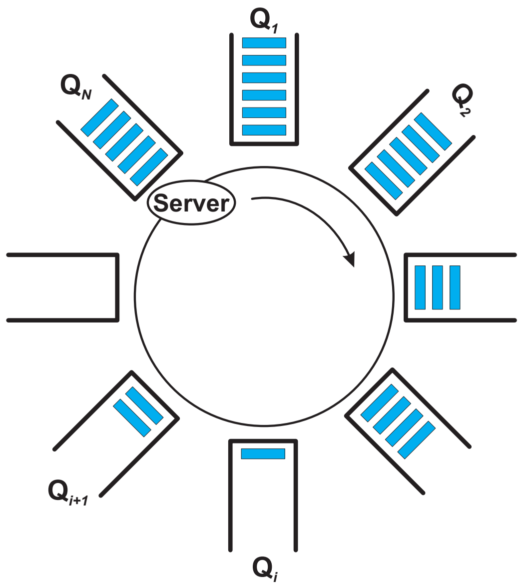
Arrival rate at queue depends on current position of the server.

Server position:

$$P \in \{V_1, S_1, \ldots, V_N, S_N\},$$

then arrivals at Q_i:

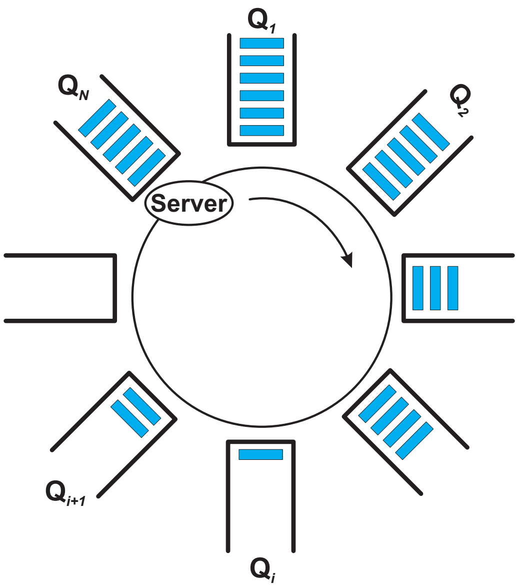
$$\text{Poisson}(\lambda_i^{(P)}).$$



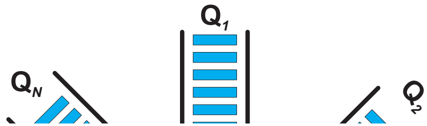
One application: arriving customers *choose* which queue to join (in a smart way), total arrival rate $\Lambda = \sum_i \lambda_i^{(P)}$ for all P.

Smart Customers: Applications

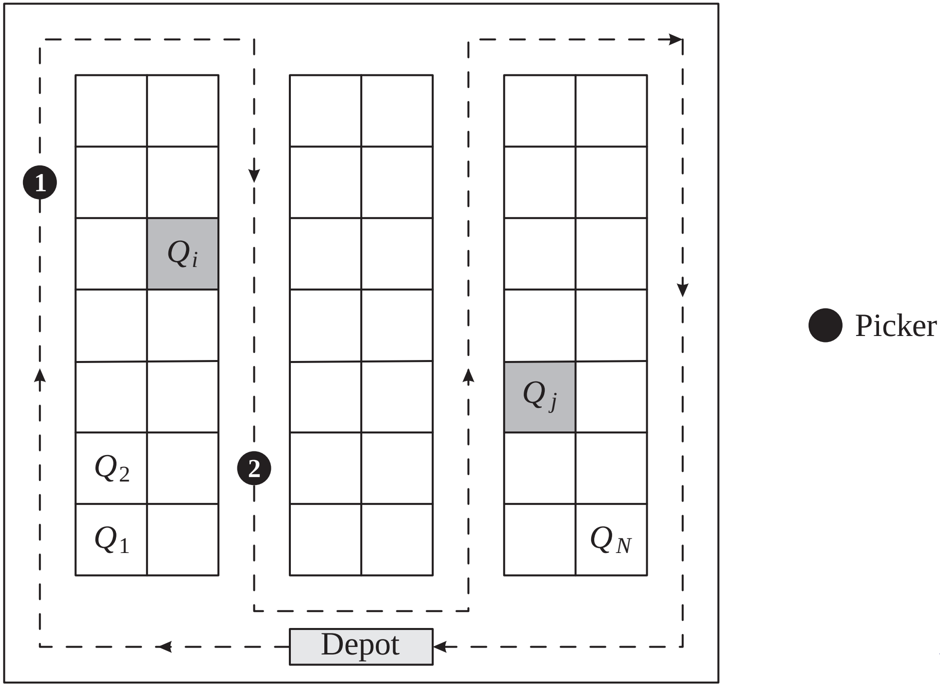
- Customers *choose* which queue to join
- Increasing arrival stream when server is approaching
- Only arrivals when server is at specific queue
- Route customers to ‘correct’ queue
  - polling tables, possibly with different disciplines
  - production system
  - dynamic order picking



Smart Customers: Applications



- Customers *choose* which queue to join
- Increasing arrival stream when server is approaching
- Only arrivals wh[...] queue
- Route customers t[...]
  - polling tables, disciplines
  - production sys[...]
  - **dynamic order picking**



Outline

- Analysis
  - Waiting time distributions
  - Mean waiting times
  - Pseudo conservation law
- Examples
  - Choosing a queue: smart (and stupid) customers
  - Routing of customers

Waiting Time Distributions

using Multi-Type Branching Processes, cf. Resing (1993)

Queue length process: N-type branching process with immigration.

Each customer present effectively replaced (i.i.d.) by random population with pgf $h_i(z_1, \ldots, z_N)$:

$$\text{gated} \quad h_i(\underline{z}) = \beta_i\Big(\sum_j \lambda_j^{(V_i)}(1 - z_j)\Big),$$

$$\text{exhaustive} \quad h_i(\underline{z}) = \pi_i\Big(\sum_{j \neq i} \lambda_j^{(V_i)}(1 - z_j)\Big).$$

We derive distributions of:

- joint queue lengths
- marginal queue lengths
- waiting times
- cycle times
- visit times
- intervisit times

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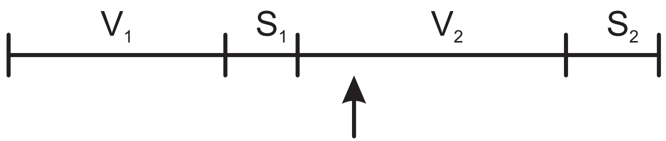
Mean Waiting Times

using Mean Value Analysis for polling models, cf. Winands et al. (2006)

$$\mathsf{E}[W_i] = \mathsf{E}[L_i]\mathsf{E}[B_i] + \rho_i \mathsf{E}[R_{B_i}] + (1-\rho_i)\mathsf{E}[T_i], \qquad \mathsf{E}[L_i] = \lambda_i \mathsf{E}[W_i],$$

$\mathsf{E}[T_i]$: mean residual time until server starts working on Q_i again.



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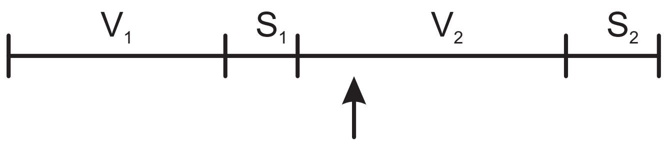
MVA adapted for Smart Customers:

$$\mathsf{E}[W_i] = \mathsf{E}[\tilde{L}_i]\mathsf{E}[B_i] + \frac{\mathsf{E}V_i}{\mathsf{E}C}\frac{\lambda_i^{(V_i)}}{\bar{\lambda}_i}\mathsf{E}[R_{B_i}] + (1 - \cdot)\mathsf{E}[T_i],$$

$$\mathsf{E}[\bar{L}_i] = \bar{\lambda}_i \mathsf{E}[W_i],$$

$$\bar{\lambda}_i = \sum_j \frac{\mathsf{E}V_j}{\mathsf{E}C}\lambda_i^{(V_j)} + \frac{\mathsf{E}S_j}{\mathsf{E}C}\lambda_i^{(S_j)}.$$



Pseudo Conservation Law

expression for $\sum_i \rho_i \mathsf{E}(W_i)$, cf. Boxma and Groenendijk (1987)

$$\sum_i \rho_i \mathsf{E}(W_i) = \rho \frac{\sum_i \rho_i \mathsf{E}(R_{B_i})}{1-\rho} + \rho\, \mathsf{E}(R_S) + \frac{\mathsf{E}(S)}{2\,(1-\rho)}\left(\rho^2 - \sum_i \rho_i^2\right) + \sum_i \mathsf{E}(Z_{ii}).$$

For smart customers, PCL only holds in two cases:

1) $\lambda_i^{(V_1)} = \lambda_i^{(V_2)} = \ldots = \lambda_i^{(V_N)} =: \lambda_i^{(V)}, \qquad i = 1, \ldots, N,$ or

2) $\sum_{i=1}^{N} \lambda_i^{(V_j)} =: \Lambda^{(V)},$ and $B_1 \stackrel{d}{=} \ldots \stackrel{d}{=} B_N, \qquad j = 1, \ldots, N,$ then …

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2) $\sum_{i=1}^{N} \lambda_i^{(V_j)} =: \Lambda^{(V)}$, and $B_1 \overset{d}{=} \ldots \overset{d}{=} B_N$, $\quad j = 1, \ldots, N$, then

$$\sum_{i=1}^{N} \overline{\rho}_i \, \mathsf{E}[W_i] = (1 - \overline{\rho}) \sum_{i=1}^{N} \frac{\mathsf{E}[S_i]}{\mathsf{E}[S]} \mathsf{E}[Y^{(S_i)}] - \sum_{i=1}^{N} \overline{\rho}_i \mathsf{E}[R_{B_i}]$$
$$+ \overline{\rho} \sum_{i=1}^{N} p_i \left(\sum_{j=1}^{N} \frac{\lambda_i^{(S_j)} \mathsf{E}[S_j]}{\sum_{k=1}^{N} \lambda_i^{(S_k)} \mathsf{E}[S_k]} \mathsf{E}[Y^{(S_j)}] + \mathsf{E}[R_{B_i}] + \frac{\rho^{(V)}}{1 - \rho^{(V)}} \mathsf{E}[R_{B^{(V)}}] \right),$$

where

$$\overline{\rho} = \overline{\lambda}_i \mathsf{E}(B_i) \text{ with } \overline{\lambda}_i = \sum_j \frac{\mathsf{E}V_j}{\mathsf{E}C} \lambda_i^{(V_j)} + \frac{\mathsf{E}S_j}{\mathsf{E}C} \lambda_i^{(S_j)}, \text{ and}$$

$$\mathsf{E}[Y^{(S_i)}] = \sum_{j=1}^{N} \left(\lambda_j^{(S_i)} \mathsf{E}[B_j] \mathsf{E}[R_{S_i}] + \mathsf{E}[Z_{jj}] \right) + \sum_{j=i+1}^{i+N-1} \sum_{k=j}^{i+N-1} \left(\lambda_j^{(S_k)} \mathsf{E}[B_j] \mathsf{E}[S_k] + \lambda_j^{(V_{k+1})} \mathsf{E}[B_j] \mathsf{E}[V_{k+1}] \right), \text{ and}$$

$$p_i = \frac{\sum_{j=1}^{N} \lambda_i^{(S_j)} \mathsf{E}[S_j] \mathsf{E}[B_i]}{\sum_{k=1}^{N} \sum_{j=1}^{N} \lambda_k^{(S_j)} \mathsf{E}[S_j] \mathsf{E}[B_k]}.$$

Choosing a queue: smart (and stupid) customers

One arrival stream: Poisson(Λ).

Which queue to join, based on server position?

Symmetric cases (exhaustive; gated): straightforward

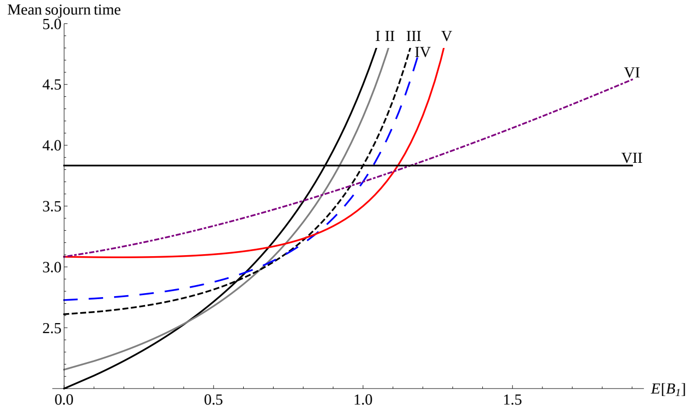
Example

Choosing a queue (ctd.)

Example for 3 queues (exhaustive): $\Lambda = 0.6$, B_2, B_3, $S_i \sim$ exp(1), and $B_1 \sim$ exp with mean $\in [0, 2]$.

Seven smart strategies:

Strategy	Queue to join during					
	V_1	S_1	V_2	S_2	V_3	S_3
I	1	1	–	1	–	1
II	1	2	1	1	–	1
III	1	2	2	1	–	1
IV	1	2	2	3	1	1
V	1	2	2	3	3	1
VI	2	2	2	3	3	1
VII	–	2	2	3	3	2



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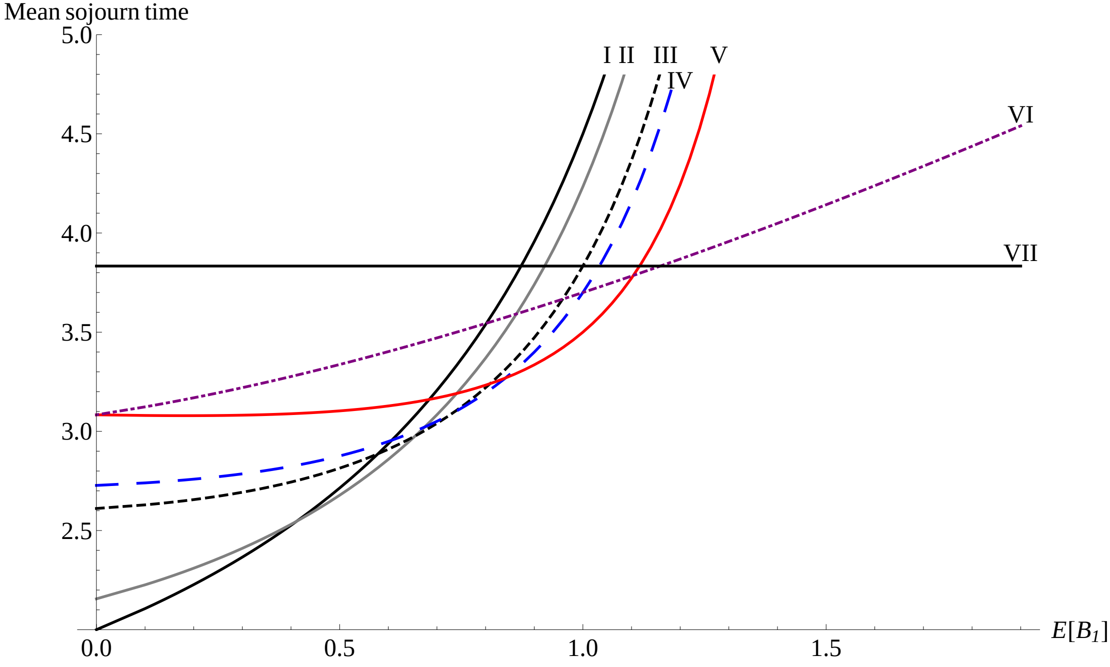
B_2, B_3, $S_i \sim \exp(1)$, and $B_1 \sim \exp$ with mean $\in [0, 2]$.

Seven smart

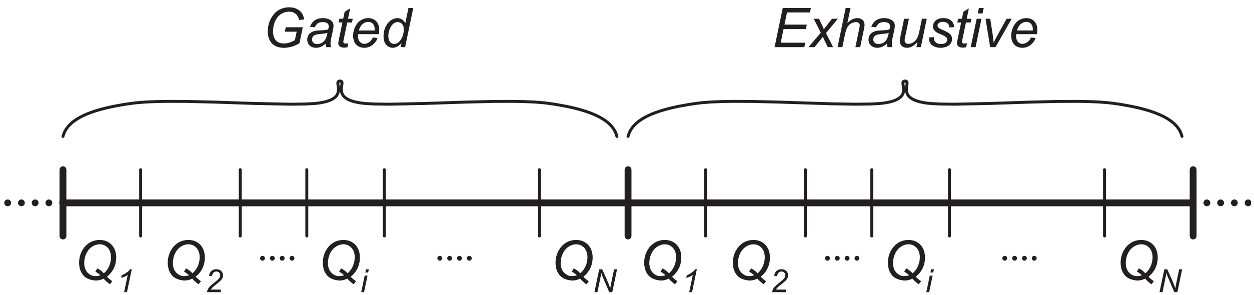
Strategy
I
II
III
IV
V
VI
VII



Example

Routing of customers

Gated/Exhaustive strategy (cf. Boxma et al. 2007)

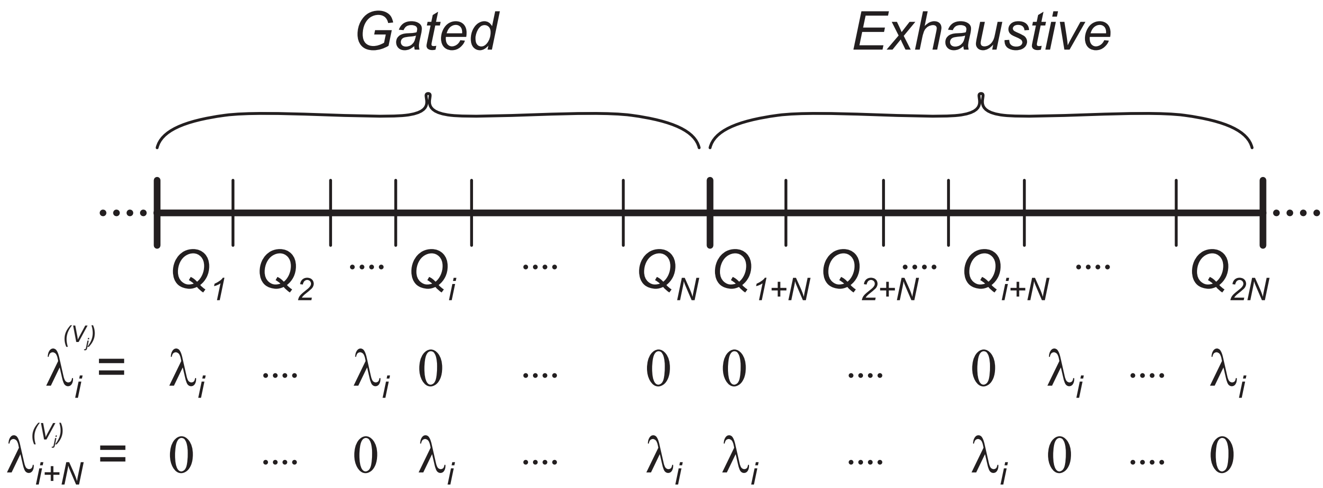


Example

Routing of customers

Gated/Exhaustive strategy (cf. Boxma et al. 2007)

Double queues, route arriving customer to correct queue:



Conclusion

Introduced **Smart Customers** for polling models

- Analysis
  - Waiting time distributions
  - Mean waiting times
  - Pseudo conservation law
- Examples
  - Choosing a queue
  - Routing of customers

M. Boon, A. van Wijk, I. Adan and O. Boxma,
A Polling Model with Smart Customers, 2010.
Available via: www.eurandom.nl/reports.